PH 5840 Lecture 12

Note Title 9/3/2015

Mixed-state entanglement

* A density operator ρ_{AB} is "separable" iff it can be realized as an ensemble of separable pure states.

i.e. ρ_{AB} is separable iff $\exists$ pure states $\{|\alpha_i\rangle\} \in \mathcal{H}_A$ and $\{|\beta_i\rangle\} \in \mathcal{H}_B$ such that,

$$\rho_{AB} = \sum_i p_i \left(|\alpha_i\rangle\langle\alpha_i|\right)_A \otimes \left(|\beta_i\rangle\langle\beta_i|\right)_B \quad \text{—} \ (*)$$

where, $\sum_i p_i = 1$.

Alternately, iff $\exists$ density operators $\{\rho_A^{(i)}\} \in \mathcal{H}_A$ and $\{\rho_B^{(j)}\} \in \mathcal{H}_B$ such that

$$\rho_{AB} = \sum_{i,j} p_{ij}\, \rho_A^{(i)} \otimes \rho_B^{(j)}, \qquad \sum_{i,j} p_{ij} = 1.$$

Equivalent, because,
$$\rho_A^{(i)} = \sum_k q_k^{(i)} |\alpha_k^{(i)}\rangle\langle\alpha_k^{(i)}|$$

$$\rho_B^{(j)} = \sum_m c_m^{(j)} |\beta_m^{(j)}\rangle\langle\beta_m^{(j)}|$$

$$\Rightarrow \rho_{AB} = \sum_{\substack{i,j \\ k,m}} p_{ij}\, q_k^{(i)} c_m^{(j)} |\alpha_k^{(i)}\rangle\langle\alpha_k^{(i)}| \otimes |\beta_m^{(j)}\rangle\langle\beta_m^{(j)}|$$

$$= \sum_{r,s} \tilde{p}_{rs} \left(|\alpha_r\rangle\langle\alpha_r|\right) \otimes \left(|\beta_s\rangle\langle\beta_s|\right)$$

Let $\sum_s \tilde{p}_{rs} |\beta_s\rangle\langle\beta_s| = |\tilde{\beta}_r\rangle\langle\tilde{\beta}_r|$

$$\langle\tilde{\beta}_r|\tilde{\beta}_r\rangle = \sum_s \tilde{p}_{rs} = b_r$$

$$\therefore \rho_{AB} = \sum_r b_r \left(|\alpha_r\rangle\langle\alpha_r|\right) \otimes \left(|\tilde{\beta}_r\rangle\langle\tilde{\beta}_r|\right)$$

same form as $(*)$

* The same density operator may be realized as an ensemble of entangled states or as an ensemble of separable states.

Eg. $\rho_{AB} = \frac{1}{4}(\mathbb{I}_A \otimes \mathbb{I}_B)$

$$= \frac{1}{4}\Big(|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{11}\rangle\langle\beta_{11}|\Big)$$

$$= \frac{1}{4}\Big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\Big)$$

- So long as there exists a decomposition in terms of product states, we say ρ_{AB} is separable/unentangled.

* Motivation for the definition:- LOCC

- Can the bipartite density operator be realized via local operations & classical communications?
- LOCC cannot increase schmidt rank from 1 to >1.
- Separable density operators can be realized via LOCC.

Peres-Horodecki Criterion / PPT condition

* Given a mixed state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, how do we check if it is separable or not?

- Consider, transpose operation:-

$$(|i\rangle\langle j|)^T = |j\rangle\langle i|$$

$$\left.\begin{aligned} M &= \sum_{i,j} m_{ij}\, |i\rangle\langle j| \\ M^T &= \sum_{ij} m_{ij}\, |j\rangle\langle i| \end{aligned}\right\} \text{Preserves trace.}$$

If $M^\dagger = M$, $\quad M^T = M^*$.

$$(\rho^T)^\dagger = (\rho^*)^\dagger = \rho^T \Rightarrow \rho^T \text{ is also Hermitian.}$$

Has the same evalues as ρ

$\therefore$ $\hat{T} : \rho \to \rho^T$ is a positive, trace-preserving map.

* However, consider the <u>partial transpose</u> operation.

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$(\hat{\mathbb{I}}_A \otimes \hat{T}_B)(\rho_{AB}) = (\hat{\mathbb{I}}_A \otimes \hat{T}_B)\,\frac{1}{2}\big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\big)$$

$$= \frac{1}{2}\big(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\big)$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Evalues: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$. $\longrightarrow$ No longer positive!!

* However, partial transpose of a separable state:

$$\rho_{AB}^{T_B} = (\hat{\mathbb{I}}_A \otimes \hat{T}_B) \sum_{i,j} p_{ij} \left(\rho_A^{(i)} \otimes \rho_B^{(j)}\right)$$

$$= \sum_{i,j} p_{ij}\, \rho_A^{(i)} \otimes \left(\rho_B^{(j)}\right)^T \geq 0$$

Still a valid density operator!

PPT criterion: If ρ_{AB} is separable, then ρ_{AB}^{PT} is non-negative.

* Converse? Consider max-entangled states:-

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |i\rangle_A |i\rangle_B . \quad \left(\text{Max. entangled in } d\text{-dimensions}\right)$$

$$\rho_{AB} = \frac{1}{d} \sum_{ij} |ii\rangle_{AB} \langle jj|_{AB}$$

$$(\hat{\mathbb{I}}_A \otimes \hat{T}_B)(\rho_{AB}) = \frac{1}{d} \sum_{ij} |ij\rangle\langle ji| = \frac{1}{d} (\text{SWAP})$$

SWAP has evalues ± 1 $\Rightarrow$ Not positive!!

SWAP :- $|\phi\rangle_A |\psi\rangle_B \longrightarrow |\psi\rangle_A |\phi\rangle_B$

$$(\text{SWAP}) \left(\sum_{ij} c_{ij} |i\rangle_A |j\rangle_B \right) = \sum_{ij} c_{ij} |j\rangle_A |i\rangle_B$$

$\therefore$ Eigenstates of SWAP are : symmetric / anti-symmetric
(+1) (-1)

$\Rightarrow$ Not a positive operator!

* PPT criterion is a necessary but not sufficient condition for separability.

* $\exists$ entangled states for which $\rho_{AB}^{PT} \geq 0$.

* Known to be necessary & sufficient for 2×2 and $2 \otimes 3$ systems.

* Multipartite entanglement :-

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

- $|GHZ\rangle$- state (named after Greenberger - Horne - Zeilinger)
- Maximally entangled tripartite state
- But separable across any bipartite cut :-

$$\rho_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$$

$$\rho_{AB} = Tr_C[\rho_{ABC}] = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) = \rho_{BC} = \rho_{AC}$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right) \quad \text{— (a)}$$

$\Rightarrow$ Mixture of separable states!

$\Rightarrow$ Not entangled across any bipartite cut.

Extra reading: {Borromean triangles and prime knots in an ancient temple", Arul. L., Resonance May 2007 Pg. 41}

* Another class of states :-

$$|\phi\rangle_{ABC} = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

- Called the $|W\rangle$-state
-

$$\sigma_{ABC} = \frac{1}{3}\Big(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100|$$
$$+ |001\rangle\langle 010| + |001\rangle\langle 100|$$
$$+ |010\rangle\langle 001| + |010\rangle\langle 100|$$
$$+ |100\rangle\langle 001| + |100\rangle\langle 010|\Big)$$

$$\sigma_{AB} = \frac{1}{3}\Big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|$$
$$+ \underbrace{|01\rangle\langle 10|} + \underbrace{|10\rangle\langle 01|}\Big) \quad \text{— (b)}$$

Not product states!

$$= \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues:

$$\lambda = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right\}$$

After partial transposition:-

$$(\hat{\mathbb{I}}_A \otimes \hat{T}_B)(\sigma_{AB}) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues:

$$\lambda = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1 \pm \sqrt{5}}{6} \right\}$$

$\therefore (\hat{\mathbb{I}}_A \otimes \hat{T}_B)(\sigma_{AB})$ has a negative eigenvalue.

$\Rightarrow \sigma_{AB}$ is not separable!

Similarly σ_{BC} & σ_{AC} are also not separable.

$\therefore$ $|W\rangle$-state & $|GHZ\rangle$-state represent two different classes of tripartite entanglement.

Note:

(i) Pure-entangled state : $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$
$|W\rangle$-state, $|GHZ\rangle$-state

(ii) Mixed entangled state: $\mathrm{Tr}_C[|W\rangle\langle W|] = \sigma_{AB}$
(see (b) above)

(iii) Pure separable state : $|\alpha\rangle \otimes |\beta\rangle$

(iv) Mixed separable state : $\mathrm{Tr}_C[|GHZ\rangle\langle GHZ|] = \rho_{AB}$
(see (a) above)